

Law of Error in Tsallis Statistics

Hiroki Suyari

*Department of Information and Image Sciences,
Faculty of Engineering, Chiba University, 263-8522, Japan**

Makoto Tsukada

*Department of Information Sciences,
Faculty of Science, Toho University, 274-8510, Japan†*

(Dated: February 2, 2008)

Abstract

Gauss' law of error is generalized in Tsallis statistics such as multifractal systems, in which Tsallis entropy plays an essential role instead of Shannon entropy. For the generalization, we apply the new multiplication operation determined by the q -logarithm and the q -exponential functions to the definition of the likelihood function in Gauss' law of error. The maximum likelihood principle leads us to finding Tsallis distribution as nonextensively generalization of Gaussian distribution.

PACS numbers: 02.50.-r, 89.70.+c

Keywords: Tsallis entropy, law of error, maximum likelihood principle, q -product

*Electronic address: suyari@ieee.org, suyari@faculty.chiba-u.jp

†Electronic address: tsukada@is.sci.toho-u.ac.jp

I. INTRODUCTION

The maximum entropy principle for *Tsallis entropy* [1][2] :

$$S_q := \frac{1 - \int f(x)^q dx}{q - 1} \quad (q \in \mathbb{R}^+) \quad (1)$$

under the constraints:

$$\int f(x) dx = 1, \quad \frac{\int x^2 f(x)^q dx}{\int f(y)^q dy} = \sigma^2 \quad (2)$$

yields the so-called *Tsallis distribution*:

$$f(x) = \frac{\exp_q(-\beta_q x^2)}{\int \exp_q(-\beta_q y^2) dy} \propto [1 + (1 - q)(-\beta_q x^2)]^{\frac{1}{1-q}}, \quad (3)$$

where $\exp_q(x)$ is the q -exponential function defined by

$$\exp_q(x) := \begin{cases} [1 + (1 - q)x]^{\frac{1}{1-q}} & \text{if } 1 + (1 - q)x > 0, \\ 0 & \text{otherwise} \end{cases} \quad (x \in \mathbb{R}) \quad (4)$$

and β_q is a positive constant related to σ and q [3][4]. For $q \rightarrow 1$, Tsallis distribution (3) recovers a Gaussian distribution. The power form in Tsallis distribution (3) has been found to be fairly fitted to many physical systems which *cannot* be systematically studied in the usual Boltzmann-Gibbs statistical mechanics [5].

The mathematical basis for Tsallis statistics comes from the deformed expressions for the logarithm and the exponential functions which are the q -logarithm function:

$$\ln_q x := \frac{x^{1-q} - 1}{1 - q} \quad (x \geq 0, q \in \mathbb{R}^+) \quad (5)$$

and its inverse function, the q -exponential function (4). Using the q -logarithm function (5), Tsallis entropy (1) can be written as

$$S_q = - \int f(x)^q \ln_q f(x) dx, \quad (6)$$

which is easily found to recover Shannon entropy when $q \rightarrow 1$.

The successful many applications of Tsallis statistics stimulate us to try to find the new mathematical structure behind Tsallis statistics [6]. Some physicists reported some relations to the q -analysis which has been usually discussed in quantum group [7][8]. Apart from the relations to the q -analysis, Borges presents a deformed algebra related to the q -logarithm and the q -exponential function which naturally emerge from Tsallis entropy [9].

Using the algebra introduced by Borges, we derive Tsallis distribution by applying the multiplication operation determined by the q -logarithm and the q -exponential function to the *maximum likelihood principle* (MLP for short). Concretely, the new multiplication operation \otimes_q , which is called the q -product in [9], is applied to the *likelihood function* in *Gauss' law of error* with the result that we obtain Tsallis distribution as nonextensively generalization of Gaussian distribution. The present derivation of Tsallis distribution is another way than the above maximum entropy principle (MEP for short, in contrast to MLP).

II. GAUSS' LAW OF ERROR

In order to clarify our ideas in the present work in section IV, we briefly review Gauss' law of error in this section.

Consider the following situation: we get n observed values:

$$x_1, x_2, \dots, x_n \in \mathbb{R} \quad (7)$$

as results of mutually independent n measurements for some observation. Each observed value x_i ($i = 1, \dots, n$) is each value of independent, identically distributed (i.i.d. for short) random variable X_i ($i = 1, \dots, n$), respectively. There exist a true value x satisfying the *additive* relation:

$$x_i = x + e_i \quad (i = 1, \dots, n), \quad (8)$$

where each of e_i is an error in each observation of a true value x . Thus, for each X_i , there exists a random variable E_i such that $X_i = x + E_i$ ($i = 1, \dots, n$). Every E_i has the same probability density function f which is differentiable, because X_1, \dots, X_n are i.i.d. (i.e., E_1, \dots, E_n are i.i.d.). Let $L(\theta)$ be a function of a variable θ , defined by

$$L(\theta) = L(x_1, x_2, \dots, x_n; \theta) := f(x_1 - \theta) f(x_2 - \theta) \cdots f(x_n - \theta). \quad (9)$$

Theorem 1 *If the function $L(x_1, x_2, \dots, x_n; \theta)$ of θ for any fixed x_1, x_2, \dots, x_n takes the maximum at*

$$\theta = \theta^* := \frac{x_1 + x_2 + \cdots + x_n}{n}, \quad (10)$$

then the probability density function f must be a Gaussian probability density function:

$$f(e) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{e^2}{2\sigma^2} \right\}. \quad (11)$$

The above result goes by the name of “*Gauss’ law of error*” [10] which is often used as an assumption in measurements in many scientific fields. In fact, on the basis of Gauss’ law of error some functions such as error function are often used to estimate error rate in measurements.

Note that the assumption (8) means each error e_i is *additive* to the true value x . This assumption essentially contributes the determination of a Gaussian probability density function as seen in the proof below. The last assumption is a representation of MLP. In MLP, the parameter θ and the function $L(\theta)$ are called *population parameter* and *likelihood function*, respectively. The statement of the last assumption is rewritten in MLP as follows: for the likelihood function $L(\theta)$ given by (9), the maximum likelihood estimator is

$$\hat{\theta} := \frac{X_1 + X_2 + \cdots + X_n}{n}. \quad (12)$$

Here we present the rigorous proof of *Gauss’ law of error*.

Proof. Taking the logarithm of the both side of the likelihood function $L(\theta)$ in (9) leads to

$$\ln L(\theta) = \ln f(x_1 - \theta) + \ln f(x_2 - \theta) + \cdots + \ln f(x_n - \theta). \quad (13)$$

Differentiating the above formula (13) with respect to θ , we have

$$\frac{L'(\theta)}{L(\theta)} = -\frac{f'(x_1 - \theta)}{f(x_1 - \theta)} - \cdots - \frac{f'(x_n - \theta)}{f(x_n - \theta)}. \quad (14)$$

The last assumption implies that when $\theta = \theta^*$ the likelihood function $L(\theta)$ takes the maximum value, so that

$$\frac{L'(\theta^*)}{L(\theta^*)} = 0 \quad (15)$$

i.e.,

$$\frac{f'(x_1 - \theta^*)}{f(x_1 - \theta^*)} + \cdots + \frac{f'(x_n - \theta^*)}{f(x_n - \theta^*)} = 0. \quad (16)$$

Let e_i^* be defined by

$$e_i^* := x_i - \theta^* \quad (i = 1, \cdots, n), \quad (17)$$

then (16) can be rewritten to

$$\sum_{i=1}^n \frac{f'(e_i^*)}{f(e_i^*)} = 0. \quad (18)$$

Using the new function $\phi(e)$ defined by

$$\phi(e) := \frac{f'(e)}{f(e)}, \quad (19)$$

(18) becomes

$$\sum_{i=1}^n \phi(e_i^*) = 0. \quad (20)$$

On the other hand, summing up the both sides of (17) for all $i = 1, \dots, n$, we obtain

$$\sum_{i=1}^n e_i^* = \sum_{i=1}^n x_i - n\theta^* = 0, \quad (21)$$

where we used (10).

Then our problem can be reduced to finding the function ϕ satisfying (20) under the constraint (21). For simplicity, “*” in e_i^* is abbreviated in the rest of this proof.

From (21) we obtain

$$\frac{de_i}{de_1} = -1 \quad (i \neq 1), \quad (22)$$

so that differentiating (20) with respect to e_1 goes to

$$\frac{d\phi(e_1)}{de_1} - \frac{d\phi(e_i)}{de_i} = 0 \quad (i \neq 1). \quad (23)$$

The choice of e_1 does not lose the lack of generality in the above two formulas (22) and (23), so in general there exists a constant $a \in \mathbb{R}$ such that

$$\frac{d\phi(e)}{de} = a. \quad (24)$$

Thus, it implies

$$\phi(e) = ae + b \quad (25)$$

for some $b \in \mathbb{R}$. Applying this formula (25) and (21) to (20), we get

$$0 = \sum_{i=1}^n \phi(e_i) = a \sum_{i=1}^n e_i + nb = nb, \quad (26)$$

so it implies

$$b = 0. \quad (27)$$

Therefore,

$$\phi(e) = ae \quad \Leftrightarrow \quad \frac{f'(e)}{f(e)} = ae. \quad (28)$$

The solution f satisfying (28) can be obtained to

$$f(e) = C \exp\left(\frac{ae^2}{2}\right) \quad (29)$$

where C is a constant. f is an even function and a probability density function, so that $a < 0$. Thus, there exists a constant $h > 0$ such that a can be set to

$$\frac{a}{2} = -h^2. \quad (30)$$

Then, f becomes

$$f(e) = \frac{h}{\sqrt{\pi}} \exp(-h^2 e^2) \quad (31)$$

where $\int f(e) de = 1$ is used. h can be defined by

$$h := \frac{1}{\sqrt{2}\sigma} > 0, \quad (32)$$

therefore we can determine f as a Gaussian probability density function with mean zero and variance σ^2 :

$$f(e) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{e^2}{2\sigma^2}\right\}. \quad (33)$$

■

Note that Gauss' law of error tells us that in measurements it is the most probable to assume a Gaussian probability distribution for *additive* noise, which is often applied to every scientific fields.

In section IV, Gauss' law of error is generalized to the nonextensive systems, which results in Tsallis distribution as a generalization of Gaussian distribution.

III. THE NEW MULTIPLICATION OPERATION DETERMINED BY q -LOGARITHM FUNCTION AND q -EXPONENTIAL FUNCTION

The new multiplication operation \otimes_q is first introduced by Borges in [9] for satisfying the following equations:

$$\ln_q(x \otimes_q y) = \ln_q x + \ln_q y, \quad (34)$$

$$\exp_q(x) \otimes_q \exp_q(y) = \exp_q(x + y). \quad (35)$$

These lead us to the definition of \otimes_q between two positive numbers

$$x \otimes_q y := \begin{cases} [x^{1-q} + y^{1-q} - 1]^{\frac{1}{1-q}}, & \text{if } x > 0, y > 0, x^{1-q} + y^{1-q} - 1 > 0 \\ 0, & \text{otherwise} \end{cases}, \quad (36)$$

which is called *q-product* in [9]. The *q-product* recovers the usual product such that $\lim_{q \rightarrow 1} (x \otimes_q y) = xy$. The fundamental properties of the *q-product* \otimes_q are almost the same as the usual product, but the distributive law does not hold in general.

$$a(x \otimes_q y) \neq ax \otimes_q y \quad (a, x, y \in \mathbb{R}) \quad (37)$$

The properties of the *q-product* can be found in [9]. For the purpose of this paper, the differentiation of the *q-logarithm* function is also needed in the next section.

$$\frac{d}{dx} \ln_q x = \frac{1}{x^q} \quad (38)$$

IV. NONEXTENSIVELY GENERALIZATION OF GAUSS' LAW OF ERROR

This section presents the law of error in Tsallis statistics with the rigorous proof along the same line as that of Gauss' law of error.

Consider the same setting as n observed values, as Gauss' law of error: we get n observed values:

$$x_1, x_2, \dots, x_n \in \mathbb{R} \quad (39)$$

as results of n measurements for some observation. Each observed value x_i ($i = 1, \dots, n$) is each value of identically distributed random variable X_i ($i = 1, \dots, n$), respectively. There exist a true value x satisfying the *additive* relation:

$$x_i = x + e_i \quad (i = 1, \dots, n), \quad (40)$$

where each of e_i is an error in each observation of a true value x . Thus, for each X_i , there exists a random variable E_i such that $X_i = x + E_i$ ($i = 1, \dots, n$). Every E_i has the same probability density function f which is differentiable, because X_1, \dots, X_n are i.i.d. (i.e., E_1, \dots, E_n are i.i.d.). Let $L_q(\theta)$ be a function of a variable θ , defined by

$$L_q(\theta) = L_q(x_1, x_2, \dots, x_n; \theta) := f(x_1 - \theta) \otimes_q f(x_2 - \theta) \otimes_q \dots \otimes_q f(x_n - \theta). \quad (41)$$

Theorem 2 *If the function $L_q(x_1, x_2, \dots, x_n; \theta)$ of θ for any fixed x_1, x_2, \dots, x_n takes the maximum at*

$$\theta = \theta^* := \frac{x_1 + x_2 + \dots + x_n}{n}, \quad (42)$$

then the probability density function f must be a Tsallis distribution:

$$f(e) = \frac{\exp_q(-\beta_q e^2)}{\int \exp_q(-\beta_q e^2) de} \quad (43)$$

where β_q is a q -dependent positive constant.

Proof. Taking the q -logarithm of the both side of the likelihood function $L_q(\theta)$ in (41) leads to

$$\ln_q L_q(\theta) = \ln_q f(x_1 - \theta) + \ln_q f(x_2 - \theta) + \cdots + \ln_q f(x_n - \theta) \quad (44)$$

where the property (34) is used. Differentiating the above formula (44) with respect to θ according to (38), we have

$$\frac{L_q'(\theta)}{(L_q(\theta))^q} = -\frac{f'(x_1 - \theta)}{(f(x_1 - \theta))^q} - \cdots - \frac{f'(x_n - \theta)}{(f(x_n - \theta))^q}. \quad (45)$$

When $\theta = \theta^*$ the likelihood function $L_q(\theta)$ takes the maximum, so that

$$\frac{L_q'(\theta^*)}{(L_q(\theta^*))^q} = 0 \quad \Leftrightarrow \quad \frac{f'(x_1 - \theta^*)}{(f(x_1 - \theta^*))^q} + \cdots + \frac{f'(x_n - \theta^*)}{(f(x_n - \theta^*))^q} = 0. \quad (46)$$

Let e_i^* and $\phi_q(e)$ be defined by

$$e_i^* := x_i - \theta^* \quad (i = 1, \dots, n), \quad \phi_q(e) := \frac{f'(e)}{(f(e))^q}, \quad (47)$$

then (46) can be rewritten to

$$\sum_{i=1}^n \phi_q(e_i^*) = 0. \quad (48)$$

On the other hand, summing up the both sides of the former formula in (47) for all $i = 1, \dots, n$, we obtain

$$\sum_{i=1}^n e_i^* = \sum_{i=1}^n x_i - n\theta^* = 0, \quad (49)$$

where we used (42).

Then our problem can be reduced to finding the function ϕ_q satisfying (48) under the constraint (49). For simplicity, “*” in e_i^* is abbreviated in the rest of this proof.

From (49) we obtain

$$\frac{de_i}{de_1} = -1 \quad (i \neq 1), \quad (50)$$

so that differentiating (48) with respect to e_1 goes to

$$\frac{d\phi_q(e_1)}{de_1} - \frac{d\phi_q(e_i)}{de_i} = 0 \quad (i \neq 1). \quad (51)$$

The choice of e_1 does not lose the lack of generality in the above two formulas (50) and (51), so in general there exists a q -dependent constant $a_q \in \mathbb{R}$ such that

$$\frac{d\phi_q(e)}{de} = a_q \Leftrightarrow \phi_q(e) = a_q e + b_q \quad (52)$$

for some $b_q \in \mathbb{R}$. Applying this formula (52) and (49) to (48), we get

$$0 = \sum_{i=1}^n \phi_q(e_i) = a_q \sum_{i=1}^n e_i + n b_q = n b_q \Rightarrow b_q = 0. \quad (53)$$

Therefore,

$$\phi_q(e) = a_q e \Leftrightarrow \frac{f'(e)}{(f(e))^q} = a_q e. \quad (54)$$

Using the property (38), (54) can be integrated with respect to e :

$$\ln_q f(e) = \frac{a_q e^2}{2} + C_q \quad (55)$$

where C_q is a q -dependent constant. Thus we obtain

$$f(e)^{1-q} = 1 + (1-q) \left(\frac{a_q e^2}{2} + C_q \right). \quad (56)$$

When $1 + (1-q) \left(\frac{a_q e^2}{2} + C_q \right) > 0$ and $1 + (1-q) C_q > 0$, $f(e)$ can be rewritten as follows:

$$f(e) = \left[1 + (1-q) \left(\frac{a_q e^2}{2} + C_q \right) \right]^{\frac{1}{1-q}} \quad (57)$$

$$= [1 + (1-q) C_q]^{\frac{1}{1-q}} \left[1 + (1-q) \frac{a_q}{2(1 + (1-q) C_q)} e^2 \right]^{\frac{1}{1-q}} \quad (58)$$

$$= \exp_q(C_q) \exp_q(-\beta_q e^2) \propto \exp_q(-\beta_q e^2) = [1 + (1-q)(-\beta_q e^2)]^{\frac{1}{1-q}} \quad (59)$$

where

$$\beta_q := \frac{-a_q}{2(1 + (1-q) C_q)}. \quad (60)$$

Thus, the normalization condition $\int f(e) de = 1$ implies that

$$\exp_q(C_q) = \frac{1}{\int \exp_q(-\beta_q e^2) de}. \quad (61)$$

Moreover, since $L_q(\theta)$ takes the maximum at $\theta = \theta^*$, $\frac{\partial^2 \ln_q L_q(\theta)}{\partial \theta^2} \Big|_{\theta=\theta^*}$ can be computed as

$$0 > \frac{\partial^2 \ln_q L_q(\theta)}{\partial \theta^2} \Big|_{\theta=\theta^*} = \frac{2n(-\beta_q)}{(\int \exp_q(-\beta_q e^2) de)^{1-q}}, \quad (62)$$

so that $\beta_q > 0$. Therefore, we can determine (43) for the probability density function f . ■

Our result $f(e)$ in (43) coincides with (3) by applying the q -product to MLP instead of MEP.

V. CONCLUSION

We obtain law of error in Tsallis statistics by applying the q -product to MLP. The q -product is defined as the new multiplication operation derived from the q -logarithm and q -exponential function. These functions, q -logarithm and q -exponential functions, naturally emerge from Tsallis entropy. Our present result reveals that in Tsallis statistics the q -product can constitute some important probabilistic formulations. The model for Tsallis distribution has been presented in [11]. Moreover, based on our result, the *error functions* in Tsallis statistics can be also formulated as similarly as those of Gauss' law of error.

-
- [1] C. Tsallis, Possible generalization of Boltzmann-Gibbs statistics, J. Stat. Phys. 52, 479-487 (1988).
 - [2] E.M.F. Curado and C. Tsallis, Generalized statistical mechanics: connection with thermodynamics, J. Phys. A 24, L69-L72 (1991); Corrigenda 24, 3187 (1991); 25, 1019 (1992).
 - [3] C. Tsallis, S.V.F. Levy, A.M.C. Souza and R. Maynard, Statistical-mechanical foundation of the ubiquity of Levy distributions in nature, Phys. Rev. Lett. 75, 3589-3593 (1995) [Erratum: 77, 5442 (1996)].
 - [4] D. Prato and C. Tsallis, Nonextensive foundation of Levy distributions, Phys. Rev. E 60, 2398-2401 (2000).
 - [5] C. Tsallis et al., Nonextensive Statistical Mechanics and Its Applications, edited by S. Abe and Y. Okamoto (Springer-Verlag, Heidelberg, 2001); see also the comprehensive list of references at <http://tsallis.cat.cbpf.br/biblio.htm>
 - [6] H. Suyari, Nonextensive entropies derived from form invariance of pseudoadditivity, Phys. Rev. E 65, 066118 (7pages) (2002); H. Suyari, On the most concise set of axioms and the uniqueness theorem for Tsallis entropy, J. Phys. A., 35, 10731-10738 (2002); H. Suyari, The uniqueness theorem for the nonextensive entropy, Proc. 2003 IEEE Int.Symp.Inform.Theory, 260 (2003).
 - [7] S. Abe, A note on the q -deformation theoretic aspect of the generalized entropies in nonextensive physics, Phys. Lett. A 224, 326-330 (1997); S. Abe, Nonextensive statistical mechanics

- of q -bosons based on the q -deformed entropy, Phys. Lett. A 244, 229-236 (1998).
- [8] R.S. Johal, q -calculus and entropy in nonextensive statistical physics, Phys. Rev. E 58, 4147-4151 (1998).
 - [9] E.P. Borges, A possible deformed algebra and calculus inspired in nonextensive thermostatics, cond-mat/0304545.
 - [10] A. Hald, A history of mathematical statistics from 1750 to 1930, New York, Wiley (1998).
 - [11] J.A. Cordeiro, M.V.B.T. Lima, R.M. Dias and F.A. Oliveira, Morphology of growth by random walk deposition, Physica A 295, 209-214 (2001).